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A canonical transformation theory of the generalized Dicke model

B V Thompson

Department of Mathematics, University of Manchester Institute of Science and Technology, PO Box 88, Manchester M60 1QD, UK

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Abstract. The Hamiltonian of the generalized Dicke model is transformed to describe collective excitations above a ground state. When the wavelength of the field modes is considerably less than the system dimensions, the condition for the existence of a phase transition in a lattice configuration differs from the long-wave result. The effect of atomic separation on certain ground state properties is examined and a short description of the excitation spectrum is given. For a dielectric crystal, it is shown that the super-radiant phase transition is impossible in principle for up to 48 modes.

1. Introduction

The Dicke model, which describes the coupling of a set of $(2S+1)$ level systems to a finite number of modes of a boson field, has been studied in connection with a variety of physical effects. These include many of the effects in non-linear optics, phenomena involving the interaction of spin waves and phonons in paramagnetic crystals (Messina and Persico 1973) and the cooperative Jahn–Teller effect in rare-earth vanadates (Elliot *et al* 1972). The latter, which is a second-order phase transition, seems to be well established experimentally, but the so called super-radiant phase transition of optics has not been observed. Recently, the theoretical basis for the existence of a phase transition in the optical case has been questioned (Rzążewski *et al* 1975). This aspect of the Dicke model is the subject of the following work. A realistic case which has not received detailed attention is the one in which the mode wavelength, while exceeding the interatomic spacing, is considerably less than the cavity dimensions. When the atoms are arranged on a crystal lattice it is found that the mathematical requirement for a phase transition to exist differs from the long-wavelength result.

Standard canonical transformation theory is applied to recast the Hamiltonian into a form which describes collective excitations above a ground state for any value of the coupling constant. The method is a rather obvious generalization of a calculation for the multi-atom model (Thompson 1975a). It displays clearly why the 'Bogolubov trick' employed by Vertogen and de Vries (1974) and Pimental and Zimerman (1975) works correctly for the ground state in this problem.

In writing the Dicke Hamiltonian in the form

$$H=\sum_k \omega_k a_k^\dagger a_k+\omega_0\sum_l \sigma_3(l)+2\sum_k\sum_l g_k(a_k\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}}+a_k^\dagger\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.\boldsymbol{l}})\sigma_1(l) \tag{1}$$

no particular spatial arrangement of the N atoms labelled by position vectors $\boldsymbol{l}$ is envisaged yet. The field modes (finite in number) associated with annihilation and creation operators a_k, $a_k^\dagger$, wavevector $\boldsymbol{k}$ and frequency ω_k, interact with the dipole moment of each atom which is represented by the spin operator $\sigma_1(\boldsymbol{l})$. The energy spacing of the atomic levels is $\hbar\omega_0$, the spin operator $\sigma_3(\boldsymbol{l})$ having eigenvalues $-S$, $-S+1, \ldots, +S$. There appears to be no loss of generality in choosing the coupling constant g_k which is $O(N^{-1/2})$ to be real and positive.

Following Duncan (1974), we have kept the anti-resonant terms of the interaction in (1). If we were to use the rotating wave approximation, a constant of the motion appears which has the effect of turning the problem into one about a finite-dimensional vector space. This is a definite disadvantage for the present approach (Thompson 1972), although interesting numerical work becomes possible when the rotating wave approximation is adopted (e.g. Nakamura and Sugano 1975).

2. The canonical transformation

Guided by the earlier calculation (Thompson 1975a) we define two unitary operators U_1 and U_2:

$$U_1 = \prod_k \exp(z_k^* a_k - z_k a_k^\dagger) \tag{2}$$

$$U_2 = \prod_l \exp \tfrac{1}{2}\theta_l(\sigma^{(+)}(l) - \sigma^{(-)}(l)). \tag{3}$$

The sets of parameters $\{\ldots z_k \ldots\}$, $\{\theta_l\}$ are at present arbitrary. New dynamical variables are now defined:

$$\begin{aligned} a_k)_{\text{new}} &= U_1 a_k U_1^\dagger \qquad a_k^\dagger)_{\text{new}} = U_1 a_k^\dagger U_1^\dagger \\ \sigma_i(l)_{\text{new}} &= U_2 \sigma_i(l) U_2^\dagger \qquad i = 1, 2, 3 \end{aligned} \tag{4}$$

and we can express the old variables in terms of the new ones as follows:

$$\begin{aligned} &a_k = a_k)_{\text{new}} - z_k; \qquad a_k^\dagger = a_k^\dagger)_{\text{new}} - z_k^* \\ &\begin{pmatrix} \sigma_1(l) \\ \sigma_3(l) \end{pmatrix} = \begin{pmatrix} \cos\theta_l & \sin\theta_l \\ -\sin\theta_l & \cos\theta_l \end{pmatrix} \begin{pmatrix} \sigma_1(l) \\ \sigma_3(l) \end{pmatrix}_{\text{new}} \\ &H\{\ldots a_k, \sigma_i(l) \ldots\} \equiv \mathcal{H}\{\ldots a_k, \sigma_i(l) \ldots\}_{\text{new}}. \end{aligned} \tag{5}$$

From now on, the 'new' label will be dropped where there is no confusion. We find that

$$\mathcal{H} = E_0 + K_1 + K_2 + K_3$$

with

$$E_0 = \sum_k \omega_k |z_k|^2 - \omega_0 S \sum_l \cos\theta_l + 2S \sum_k \sum_l g_k \sin\theta_l (z_k \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}} + \text{CC}) \tag{6}$$

$$\begin{aligned} -K_1 = &\sum_k \omega_k (z_k a_k^\dagger + \text{HC}) + \omega_0 \sum_l \sin\theta_l \sigma_1(l) \\ &+ 2\sum_k \sum_l g_k (S a_k \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}} \sin\theta_l + \text{HC} + z_k \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{l}} \cos\theta_l \sigma_1(l) + \text{HC}) \end{aligned} \tag{7}$$

$$K_2 = \sum_k \omega_k a_k^\dagger a_k + \omega_0 \sum_l \cos\theta_l (S + \sigma_3(l)) + 2\sum_k \sum_l g_k [a_k \cos\theta_l \, e^{ik.l} \sigma_1(l) + \text{HC}$$

$$+ (-z_k) \, e^{ik.l} \sin\theta_l (\sigma_3(l) + S) + \text{HC}] \tag{8}$$

$$K_3 = 2\sum_k \sum_l g_k (a_k \, e^{ik.l} + \text{HC}) \sin\theta_l (S + \sigma_3(l)). \tag{9}$$

It will be seen that a number of terms linear in S have been included in the above equations although they must cancel in $\mathcal{H}$. The reason for their appearance can be understood by considering the associated variational problem. If we denote the state of no photons and all atoms in their lowest states by ϕ_0, then a suitable ground state trial function ψ for H is given by

$$\psi = U_1 U_2 \phi_0. \tag{10}$$

An upper bound for the ground state energy of H is then

$$\langle \phi_0 | U_2^\dagger U_1^\dagger H U_1 U_2 | \phi_0 \rangle$$

which is the same as E_0 in (6). Furthermore, in this trial ground state ψ, the expectation value of $\sigma_3(l)$ is $-S \cos\theta_l$, or equivalently, the expectation value of $\sigma_3(l)_{\text{new}}$ in the transformed ground state is $-S$. It is vital that K_1 should vanish identically if K_2 is to describe collective excitations above the new ground state. By choosing the parameters z_k, θ_l which make E_0 stationary this is assured. The stationarity of E_0 requires

$$\omega_k z_k^* + 2 g_k S \sum_l e^{ik.l} \sin\theta_l = 0 \tag{11}$$

$$\omega_0 \sin\theta_l + 2\cos\theta_l \sum_k g_k (z_k \, e^{ik.l} + \text{CC}) = 0. \tag{12}$$

The number of photons present in state ψ is $\Sigma_k |z_k|^2$ and so the existence of non-trivial solutions of (11) and (12) points to the existence of the super-radiant phase change. In terms of the θ, the photon number n (for modes of the same frequency ω_k) is

$$n = \tfrac{1}{2} S \omega_0 \omega_k^{-1} \sum_l (\sec\theta_l - \cos\theta_l) \tag{13}$$

and in general,

$$E_0 = -\tfrac{1}{2} S \omega_0 \sum_l (\sec\theta_l + \cos\theta_l) \leqslant -S\omega_0 N \tag{14}$$

showing the instability of the vacuum state ($z_k = 0$, $\theta_l = 0$).

Before we attempt to solve (11), (12) for special configurations of the atoms, two general comments may be made. Upon elimination of the θ variables followed by the application of some simple inequalities on complex numbers, we easily find

$$|z_k| \leqslant 4 g_k S N \omega_0^{-1} \omega_k^{-1} \sum_q g_q |z_q|$$

which leads to

$$8 S N \omega_0^{-1} \sum_k g_k^2 \omega_k^{-1} \geqslant 1 \tag{15}$$

as a necessary condition for non-zero z_k to exist. The equality in (15) applies if all the atoms are located at one site, that is to say, if we are in the very long-wave limit (see e.g. Hioe 1973).

If instead the z_k are eliminated we get

$$\tan\theta_l = 8S\omega_0^{-1}\sum_k\sum_{l'} g_k^2\omega_k^{-1}\cos \boldsymbol{k}\cdot(\boldsymbol{l}-\boldsymbol{l}')\sin\theta_{l'} \tag{16}$$

which is formally similar to the BCS integral equation in the Anderson (1958) approach. If a solution set $\{\theta_l\}$ is found then $\{-\theta_l\}$ is also a solution. This sign change corresponds to a grand reversal of the x component of 'spins' of the Hamiltonian H which in turn implies a reversal of the macroscopic dipole moment of the ground state. As in the theory of ferromagnetism we assume that a transition from 'all spins up' to 'all spins down' has a vanishingly small probability so that interference of these degenerate ground states may be ignored. A situation where tunnelling may be important has been discussed recently for the multi-atom model (Messina and Persico 1975).

The final step in the transformation of H is to express the new spin operators $\sigma_i(l)_{\text{new}}$ in terms of boson operators b_l, $b_l^\dagger$ using the formula of Holstein and Primakoff (1940):

$$\begin{aligned}&\sigma_3(l)_{\text{new}} = -S + b_l^\dagger b_l\\ &\sigma^{(+)}(l)_{\text{new}} = b_l^\dagger(2S - b_l^\dagger b_l)^{1/2} \qquad \sigma^{(-)}(l)_{\text{new}} = (2S - b_l^\dagger b_l)^{1/2} b_l.\end{aligned} \tag{17}$$

The assumption now made is that for the low-lying excited states of $\mathcal{H}$ the expectation value of the number operator $b_l^\dagger b_l$ is very much smaller than S. The similarity to the procedure in spin-wave theory is obvious and we shall consider later the case where the vectors $\boldsymbol{l}$ define a crystal lattice.

Using the approximations

$$\sigma^{(+)}(l)_{\text{new}} \simeq (2S)^{1/2} b_l^\dagger; \qquad \sigma^{(-)}(l)_{\text{new}} \simeq (2S)^{1/2} b_l$$

of (17) we find

$$K_2 = \sum_k \omega_k a_k^\dagger a_k + \omega_0 \sum_l \sec\theta_l\, b_l^\dagger b_l + (2S)^{1/2}\sum_l\sum_k g_k \cos\theta_l (b_l + b_l^\dagger)(a_k\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{l}} + \text{HC}) \tag{18}$$

$$K_3 = 2\sum_k\sum_l g_k \sin\theta_l\, b_l^\dagger b_l (a_k\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{l}} + \text{HC}) \tag{19}$$

where the θ_l are solutions of (11), (12).

3. The ground state

In this and the following sections we take the case of an atomic arrangement with a centre of inversion symmetry at $\boldsymbol{l} = \boldsymbol{0}$ so that

$$\theta_l = \theta_{-l}. \tag{20}$$

From (11), $z_k = z_{-k}^* = z_k^*$ and so the imaginary part of z_k vanishes. Denoting the real part of z_k by x_k we can write (11), (12) in the form:

$$\begin{aligned}&\omega_k x_k + 2Sg_k\sum_l \cos\boldsymbol{k}\cdot\boldsymbol{l}\,\sin\theta_l = 0\\ &\omega_0\tan\theta_l + 4\sum_k g_k x_k \cos\boldsymbol{k}\cdot\boldsymbol{l} = 0.\end{aligned} \tag{21}$$

These equations are readily solved when two field modes of equal but opposite wavevector are involved, e.g. $\boldsymbol{k}$ and $-\boldsymbol{k}$. For non-trivial solutions we require:

$$\frac{16g_k^2SN}{\omega_0\omega_k}\left(\frac{1}{N}\sum_l \frac{\cos^2\boldsymbol{k}.\boldsymbol{l}}{(1+\epsilon^2\cos^2\boldsymbol{k}.\boldsymbol{l})^{1/2}}\right)=1 \tag{22}$$

$$\epsilon^2 \equiv 64g_k^2x_k^2\omega_0^{-2}=\mathrm{O}(1) \tag{23}$$

which is essentially the zero temperature limit of a one-mode result for Hioe's model B (1973, equation (39)). Since the maximum value of the sum in parenthesis in (22) occurs for $\epsilon=0$, the necessary and sufficient condition for solutions to exist is

$$16g_k^2S\omega_0^{-1}\omega_k^{-1}\sum_l \cos^2\boldsymbol{k}.\boldsymbol{l}>1 \tag{24}$$

and for short waves the sum in (24) has the value $\frac{1}{2}N$ in condensed matter. Imagine a rectangular block of material as working substance so that periodic boundary conditions on the field modes give an integral number of $\frac{1}{2}$-waves inside it. The sum in (23) may now be written in terms of complete elliptic integrals E, K to modulus κ where

$$\kappa^2 \equiv \epsilon^2/(1+\epsilon^2)$$

and (22) becomes

$$32g_k^2SN\omega_0^{-1}\omega_k^{-1}\pi^{-1}(1+\epsilon^2)^{-1/2}B(\kappa)=1 \tag{25}$$

with $\kappa^2B(\kappa)\equiv E-(1-\kappa^2)K$. The function $B(\kappa)$ (Jahnke and Emde 1945) decreases montonically from $\pi/4$ and tends to zero as ϵ increases, making the solution of (25) for given g_k relatively straightforward. Expressions for the ground state energy E_0 and total photon number are:

$$E_0=-N\omega_0S\pi^{-1}(1+\epsilon^2)^{1/2}[E+(1-\kappa^2)K]$$

$$n=N\omega_0S\pi^{-1}\omega_k^{-1}(1+\epsilon^2)^{1/2}[E-(1-\kappa^2)K].$$

The field intensity, polarization and atomic inversion averaged over a cell in the neighbourhood of site $\boldsymbol{l}$ are readily seen to be proportional to $\tan\theta_l$, $\sin\theta_l$ and $\cos\theta_l$ respectively. Now

$$\sin\theta_l=\epsilon\cos\boldsymbol{k}.\boldsymbol{l}(1+\epsilon^2\cos^2\boldsymbol{k}.\boldsymbol{l})^{-1/2} \tag{26}$$

and while the right-hand side of (26) is a periodic function of $\boldsymbol{l}$, it is not sinusoidal. In the strong coupling regime where ϵ is large, the value of $\sin\theta_l$ changes from $+\kappa$ to $-\kappa$ in the space of about ϵ^{-1} of the wavelength, pointing to a sort of domain structure (figure 1) in the ground state.

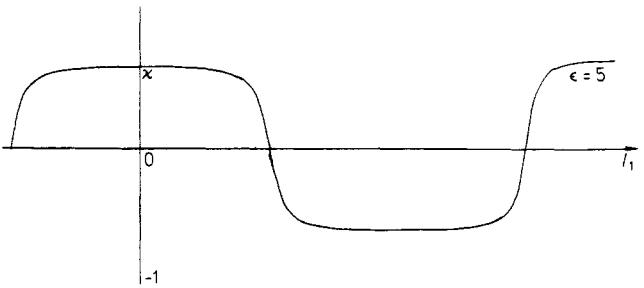

Figure 1. Schematic behaviour of polarization in the optical model; $\boldsymbol{k}$ in 1-direction.

4. Low-lying energy levels in the case of a crystal lattice

The purpose of this section is to diagonalize the coupled oscillator Hamiltonian K_2 of (18) but we verify first that the anharmonic term K_3 given by (19) can be treated as a small perturbation on the excited states near the ground state. This is easily done in terms of spin-wave variables B_q rather than the b_l. We set

$$b_l = N^{-1/2} \sum_q B_q \, e^{i\boldsymbol{q}.\boldsymbol{l}} \tag{27}$$

where $\{\boldsymbol{q}\}$ are the set of wavevectors appropriate for a lattice with periodic boundary conditions. The operators B_q, $B_q^\dagger$ satisfy the usual boson commutation relations. From (26), $\sin\theta_l$ may be written as a Fourier series using the cosines of odd multiples of $\boldsymbol{k}.\boldsymbol{l}$ i.e.:

$$\sin\theta_l = \sum_{n=0}^{\infty} A_{2n+1} \cos(2n+1)\boldsymbol{k}.\boldsymbol{l}. \tag{28}$$

Upon substitution of (27) and (28) into the anharmonic term we obtain

$$K_3 = \sum_q \sum_k \sum_n g_k A_{2n+1} (a_k B_q B^\dagger_{q+(2n+2)k} + a_k B_{q+2nk} B_q^\dagger + \text{HC}). \tag{29}$$

The numerical factors in each term are proportional to $N^{-1/2}$. Other terms involving sums over q in (18) do not contain this factor $N^{-1/2}$. It appears that the cubic term vanishes in the thermodynamic limit but for finite N it can only correct the energy levels to order N^{-1}.

Examination of the interaction term in (18) using the Fourier expansion of $\cos\theta_l$, shows that spin waves of wavevectors $\pm\boldsymbol{k}$, $\pm 3\boldsymbol{k}$, $\pm 5\boldsymbol{k}$ etc., are coupled to field modes $\boldsymbol{k}$, but that the strength of the interaction rapidly diminishes for the higher harmonics. Thus spin waves of optical wavelength are mainly concerned, and for these it is known that the corrections arising from use of the full Holstein–Primakoff formula are not very important.

We write down the Heisenberg equations of motion for the operators a_k, b_l, $a_k^\dagger$, $b_l^\dagger$ e.g.:

$$\dot{a}_k = \mathrm{i}[K_2, a_k] \tag{30}$$

and then argue that there exist suitable linear combinations of operators:

$$\alpha = \sum_k (c_{1k} a_k + c_{2k} a_k^\dagger) + \sum_l (c_{3l} b_l + c_{4l} b_l^\dagger) \tag{31}$$

such that

$$\dot{\alpha} = -\mathrm{i}\lambda\alpha; \qquad \dot{\alpha}^\dagger = \mathrm{i}\lambda\alpha^\dagger \tag{32}$$

in which λ is an eigenvalue. Combining (30), (31) and (32) gives a set of homogeneous equations for the coefficients in (31) and hence a secular determinant for λ. If we carry out this procedure but eliminate the coefficients c_{3l}, c_{4l} we obtain, for modes of equal frequency:

$$(\lambda^2 - \omega_k^2)(c_{1k} - c_{2-k}) = \sum_q T_\lambda(\boldsymbol{k}-\boldsymbol{q})(c_{1q} - c_{2-q}) \tag{33}$$

with

$$T_\lambda(\boldsymbol{k}-\boldsymbol{q}) \equiv 8g_k^2 S\omega_0\omega_k \sum_l \frac{\cos\theta_l \cos(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{l}}{\lambda^2-\omega_0^2\sec^2\theta_l}. \tag{34}$$

In the case of two modes $\boldsymbol{k}$, $-\boldsymbol{k}$ the secular determinant gives the equations:

$$\lambda^2-\omega_k^2 = T_\lambda(0) \pm T_\lambda(2\boldsymbol{k}). \tag{35}$$

A characteristic of the cooperative phase transition is the existence here of a root λ such that $\lambda \to 0$ as $8g_k^2 SN\omega_0^{-1}\omega_k^{-1} \to 1$. The number of distinct denominators in the summand of (34) is large at optical wavelengths so that $T_\lambda(\boldsymbol{q})$ considered as a function of λ^2 exhibits asymptotes densely distributed between ω_0^2 and $\omega_0^2(1+\epsilon^2)$ as in figure 2. There, the straight line represents the left-hand side of (35), and the curve is $T_\lambda(0)+T_\lambda(2k)$.

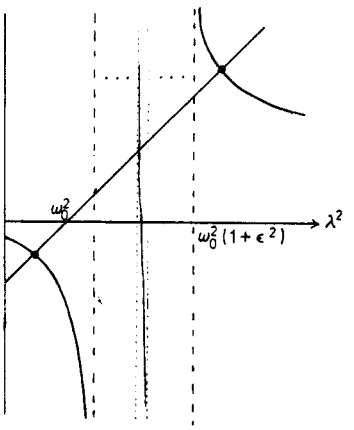

Figure 2. Roots of the dispersion relation (35).

Perhaps the only remarkable feature is the presence of the isolated roots, one well above and one well below the quasi-continuous band between ω_0 and $\omega_0(1+\epsilon^2)^{1/2}$. They may be found numerically with the aid of the complete elliptic integral of the third kind. Remembering the lower sign in (34), we have, in all, four isolated roots in this case. The roots in the band are essentially given by the values of $\omega_0 \sec\theta_l$ and can be interpreted as a kind of Stark shift for $\theta_l < 45°$, since

$$\omega_0 \sec\theta_l \simeq \omega_0 + \tfrac{1}{2}\omega_0 \tan^2\theta_l + \ldots$$

and the second term is proportional to the square of the field intensity at site $\boldsymbol{l}$.

There will be a correction to E_0, the ground state energy, arising from the change in zero point energy when we transform from the a_k, b_l variables to the α variables. This is related by the residue theorem to a suitable contour integral:

$$-(2\pi i)^{-1}\oint \ln\left(1-\frac{T_\lambda(0)\pm T_\lambda(2k)}{\lambda^2-\omega_k^2}\right) d\lambda$$

but we shall not attempt to develop this. It is clear from figure 2 that the answer must be O(1) as it was in the multi-atom case.

5. The critical condition in a crystal lattice

We consider a set of modes of the same frequency with wavevectors $\pm\boldsymbol{k}_1, \pm\boldsymbol{k}_2, \pm\boldsymbol{k}_3 \ldots$ which form a 'star', i.e. all members of the set may be reached by applying the symmetry operations of the crystal point group to any one of the vectors. For a general direction of $\boldsymbol{k}_1$ say, the number of vectors M in the star corresponds to the order of the group but it will be a factor of M if $\boldsymbol{k}_1$ is parallel to a symmetry axis. For the simple cubic lattice, $M = 48$.

In developing (21) we note that

$$x_{\pm k_1} = x_{\pm k_2} = \ldots \equiv x$$

$$\omega_{\pm k_1} = \omega_{\pm k_2} = \ldots \equiv \omega$$

$$g_{\pm k_1} = g_{\pm k_2} = \ldots \equiv g$$

and so if $x \neq 0$,

$$\omega = 8g^2 S\omega_0^{-1} \sum_l \cos \boldsymbol{k} \cdot \boldsymbol{l} \frac{2\Sigma_q' \cos \boldsymbol{q} \cdot \boldsymbol{l}}{[1+\epsilon^2(\Sigma_q' \cos \boldsymbol{q} \cdot \boldsymbol{l})^2]^{1/2}}. \tag{36}$$

The prime in the q-sums indicates that one of the pair $(\boldsymbol{k}_i, -\boldsymbol{k}_i)$ is included. Summing (36) over these $\frac{1}{2}M$ modes gives

$$\tfrac{1}{2}M = 16g^2 S\omega_0^{-1}\omega^{-1} \sum_l Q^2(1+\epsilon^2 Q^2)^{-1/2}$$

with $Q = \Sigma_q' \cos \boldsymbol{q} \cdot \boldsymbol{l}$. Hence

$$\tfrac{1}{2}M < 16g^2 S\omega_0^{-1}\omega^{-1} \sum_l Q^2. \tag{37}$$

Now

$$Q^2 = \sum_i \cos^2 \boldsymbol{k}_i \cdot \boldsymbol{l} + \sum_{i>j} \cos(\boldsymbol{k}_i - \boldsymbol{k}_j) \cdot \boldsymbol{l} + \cos(\boldsymbol{k}_i + \boldsymbol{k}_j) \cdot \boldsymbol{l}$$

and so for periodic boundary conditions $\Sigma_l Q^2 = \frac{1}{4}NM$ giving finally

$$8SNg^2\omega_0^{-1}\omega^{-1} > 1 \tag{38}$$

which is independent of M in contrast to (15).

6. Conclusion

The underlying idea of this analysis is that if the ground state of the Dicke model does not consist of a mixed state of excited atoms and photons in macroscopic numbers, then no phase transition can occur at some finite temperature. We have shown that the condition (15) quoted by Hioe (1973) as the requirement for the existence of a phase transition when spatial dependence is neglected, is a necessary condition in the usual mathematical sense, and its being satisfied does not guarantee that a transition is possible. For many modes of equal frequency in a crystal lattice the condition (15) is replaced by (38) as discussed in § 5. It is more difficult to investigate what happens when the mode wavevectors do not form a star or when the symmetry is broken in another way.

We ought to ask if it is possible in principle for (38) to be satisfied for the optics case. On substituting appropriate parameters for a dielectric model of two-level atoms into (38) (Kittel 1963, Thompson 1975b) the result is

$$1-\epsilon_0^{-1}>1$$

where ϵ_0 is the static dielectric constant. This inequality cannot be satisfied and we are forced to conclude that in optics no Dicke-type transition can occur.

One feature of the theory arising inevitably from the restriction of the k-sum in (1) to a finite set of field modes, is the 'domain' structure of the ground state in a crystal. No effects attributable to such a structure have been reported for the cooperative Jahn–Teller effect.

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